1. Amplitude Modulation **(3 pts)**

The non-periodic oscillation,

$$E(t) = E_0\left(1+\alpha\cos(\omega_m t)\right)\cdot\cos(\omega_c t)$$

describes a carrier frequency ω_c that is amplitude-modulated by a cosine of frequency ω_m. Show that this expression is equivalent to the superposition of three waves with frequencies ω_c and $(\omega_c \pm \omega_m)$. The frequencies $\omega_c + \omega_m$ and $\omega_c - \omega_m$, respectively, constitute the upper and lower sidebands and define the bandwidth needed to transfer signals in AM. Using this definition of the sidebands, explain what bandwidth is needed to transmit the full audible range in a radio channel.

$$\begin{aligned}E(t) &= E_0\cos(\omega_c t) + \alpha E_0\cos(\omega_m t)\cos(\omega_c t)\\ &= E_0\cos(\omega_c t) + \frac{\alpha E_0}{2}\left[\cos\left((\omega_c-\omega_m)t\right)+\cos\left((\omega_c+\omega_m)t\right)\right]\end{aligned}$$

In the range, 20 Hz < ν_{audio} < 20 kHz, the maximum modulation frequency needed is obviously $\Delta\nu$ = 20 kHz. The required frequency (ν) bandwidth is then 2 $\Delta\nu$ = 40 kHz.

2. Wave Propagation in Periodic Medium **(3 pts)**

Determine the phase and group velocities of a wave propagating in a periodic structure where

$$\omega(k) = 2\omega_0\sin(kl/2)$$

Write v_{ph} in terms of sinc(kl/2).

$$v_{ph} = \frac{\omega}{k} = \frac{2\omega_0}{k}\sin(kl/2) = \frac{l}{2}\frac{2\omega_0}{kl/2}\sin(kl/2) = \omega_0 l\,\mathrm{sinc}(kl/2)$$

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = 2\omega_0\cdot\frac{l}{2}\cdot\cos(kl/2) = \omega_0 l\cos(kl/2)$$

3. Gaussian Wavetrain (6 pts)

In complex notation, compute the Fourier transform of a Gaussian wave packet:

$$f(x) = \tfrac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \cdot \cos(k_p x)$$

using the standard integral $\int_0^\infty \mathrm{e}^{-ax^2}\cos(bx)\,\mathrm{d}x = \sqrt{\pi/4a}\cdot \mathrm{e}^{-b^2/4a}$.

$$A(k) = \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \cdot \cos(k_p x)\mathrm{e}^{-ikx}\mathrm{d}x = \int_{-\infty}^{\infty} \tfrac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} \cdot \tfrac{1}{2}(\mathrm{e}^{ik_p x} + \mathrm{e}^{-ik_p x})\mathrm{e}^{-ikx}\mathrm{d}x$$

$$= \tfrac{1}{2\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \left(\mathrm{e}^{-i(k-k_p)x} + \mathrm{e}^{-i(k+k_p)x}\right)\mathrm{d}x$$

$$= \tfrac{1}{2\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} \left(\cos\big((k-k_p)x\big) + \mathrm{i}\sin\big((k-k_p)x\big) + \cos\big((k+k_p)x\big) + \mathrm{i}\sin\big((k+k_p)x\big)\right)\mathrm{d}x$$

Only the real part of the integral is relevant, therefore, with $a = 2\sigma^2$ and $b = k \pm k_p$:

$$A(k) = \int_0^\infty \mathrm{e}^{-ax^2}\left(\cos(b_+ x) + \cos(b_- x)\right)\mathrm{d}x = \tfrac{1}{2\sqrt{2\pi}\sigma}\sqrt{\tfrac{\pi}{1/2\sigma^2}}\left(\mathrm{e}^{-\frac{1}{2}(k-k_p)^2\sigma^2} + \mathrm{e}^{-\frac{1}{2}(k+k_p)^2\sigma^2}\right) \text{ where}$$

$A(k) = \tfrac{1}{2}\mathrm{e}^{-\frac{1}{2}(k-k_p)^2\sigma^2}$ is the physically relevant solution around positive values of k near k_p.

4. Coherence Length of a Wavetrain (4 pts)

Derive an expression for the vacuum coherence length of a wavetrain with a frequency bandwidth $\Delta\nu$ as a function of linewidth $\Delta\lambda$ and mean wavelength λ_0.

$\Delta l_c = c\Delta t_c = c/\Delta\nu$. In vacuum, $\dfrac{\omega_0}{k_0} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k} = c$, and therefore $\Delta\nu = \nu_0 \cdot \dfrac{\Delta\lambda}{\lambda_0}$

Then, $\Delta l_c = \dfrac{c\lambda_0}{\Delta\lambda\nu_0} \approx \dfrac{\lambda_0^2}{\Delta\lambda}$.

5. Atomic Transition **(4 pts)**

A visible photon is emitted in an atomic transition during $\Delta t = 10^{-8}$ s. How long is the wave packet? From the result of problem 4, estimate the linewidth of the wavetrain at $\lambda_0 = 500$ nm.

What is the relative frequency stability?

$$\Delta l_c = c\Delta t_c = 3 \text{ m}$$

$$\Delta\lambda \approx \frac{\lambda_0^2}{\Delta l_c} = 8\cdot 10^{-5} \text{ nm}$$

$$\frac{\Delta\nu}{\nu_0} = \frac{\Delta\lambda}{\lambda_0} \approx 1.6\cdot 10^{-7}$$